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LETTER TO THE EDITOR

A solvable hierarchy of $(N+1)$-dimensional non-linear evolution equations with constraints

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Abstract. Matrix-multipolar asymptotic modules are used for introducing and solving a hierarchy of systems of purely differential non-linear equations. Each system consists of $(N+1)$-dimensional evolution equations and of quite strong differential constraints. Specific constructions of matrix-multipolar asymptotic modules are given, in particular $\bar{\partial}$ constructions.

In previous papers [1-4] we used the concept of the asymptotic module (AM) as a basic tool for introducing and solving systems of non-linear equations (NE). Generally speaking an AM is a set of matrix-valued functions $\varphi(k, x, t)$ where $x=(x_1, \ldots, x_N)$ with a differentiable structure in $(x, t) \in \mathbb{R}^N \times \mathbb{R}$ and an 'asymptotic' analytic structure in k around some poles $k_1, \ldots, k_N$ on the Riemann sphere S. These two structures are coupled thanks to a $\mathscr{D}$ module structure where $\mathscr{D}$ is some ring of linear differential operators involving $\partial_l = \partial/\partial x_l$ $(1 \leq l \leq N)$, $\partial_t = \partial/\partial t$ and rational coefficients in k. Then some system of NE connecting some 'asymptotic' coefficients depending on x and t may arise as a compatibility condition between the differentiable and the analytic structures. AM of the 'same type' lead to the introduction of the same system for which each of them affords one solution. In 'good' cases these systems are purely differential and involve a small number of unknown functions. This scheme is developed in various contexts for $N=2$ in [1-4].

In this letter we will give a precise definition of a special kind of AM called the matrix-multipolar AM (MMAM) which involves 2×2 matrix-valued functions with N poles for general $N \geq 1$. We will give specific constructions of MMAM (in particular $\bar{\partial}$ constructions) and we will show how MMAM allow us to introduce systems of purely differential NE and to determine explicit classes of their solutions. The case $N=1$ corresponds to the usual AKNS hierarchy [5] and the case $N=2$ agrees with [1, 3].

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Each system consists of $(N+1)$-dimensional evolution NE and of quite strong differential non-linear constraints. Among these systems we mention

$$\partial_t q_l = 2\mathrm{i}A_l[-\tfrac{1}{4}(\partial_l)^2 q_l + \tfrac{1}{2}q_l|q_l|^2] + \left(\sum_{m\neq l}\frac{2\mathrm{i}A_m}{(k_l-k_m)^2}\right)q_l - \sum_{m\neq l}\frac{2A_m}{k_l-k_m}D_m(l,m)q_l$$
$$+\sum_{m\neq l}\frac{2\mathrm{i}A_m}{(k_l-k_m)^2}\frac{D_m(l,m)q_l}{D_l(m,l)q_m}\left(-\tfrac{1}{2}q_m\frac{(k_l-k_m)^2}{8}D_m(l,m)D_l(m,l)q_m\right)$$
$$+\sum_{m\neq l}\frac{2\mathrm{i}A_m}{(k_l-k_m)^2}\frac{D_m(l,m)q_l}{D_l(l,m)\bar{q}_m}\left(-\tfrac{1}{2}\bar{q}_m+\frac{(k_l-k_m)^2}{8}D_m(m,l)D_l(l,m)\bar{q}_m\right) \tag{1a}$$

$$|D_l(m,l)q_m| = |D_m(l,m)q_l| \qquad l\neq m \tag{1b}$$

$$\frac{D_l(m,l)D_m(l,m)q_l}{q_l} = \frac{D_l(l,m)D_m(m,l)\bar{q}_l}{\bar{q}_l} = \frac{D_l(m,l)D_m(l,m)q_m}{q_m} \qquad l\neq m \tag{1c}$$

$$\frac{(k_l-k_m)^2}{16}\left(\frac{D_l(m,l)D_m(l,m)q_l}{q_l}\right)^2 - 1 = \frac{(k_l-k_m)^2}{4}|D_m(l,m)q_l|^2 \qquad l\neq m \tag{1d}$$

where the unknowns are the complex functions $q_l(x,t)$ $(l=1,\ldots,N)$, overbars mean complex conjugates, A_l and $k_l(l=1,\ldots,N)$ are given real numbers with $k_l\neq k_{l'}$ if $l\neq l'$, and $D_l(m,n) \doteq \partial_l + 2\mathrm{i}/(k_m-k_n)$. We remark that the system $(1a)$–$(1d)$ describes a time evolution in the manifold of solutions to the N-dimensional NE $(1b)$–$(1d)$. Note that for $A_{l_0}=-2$ and $A_l=0$ if $l\neq l_0$, $(1a)$ becomes the non-linear Schrödinger equation $\partial_t q_{l_0} - \mathrm{i}(\partial_{l_0})^2 q_{l_0} + 2\mathrm{i}q_{l_0}|q_{l_0}|^2 = 0$ relative to the variables (x_{l_0}, t_0). For $N=2$, the system $(1a)$–$(1d)$ can be reduced to the system $(2a^+)$, $(2b^+)$ and $(2c^+)$ of [3] by means of elementary transformations.

In this letter proofs are only sketched. A full account of our work will be published elsewhere with additional results.

In order to define a MMAM we introduce the following notation.

$\{k_l\}_1^N$ is a set of N distinct complex numbers.

$U=\{k\}$ is a subset of the Riemann sphere S admitting $\{k_l\}_1^N$ and ∞ as boundary points.

$\Omega=\{(x,t)$ where $x=(x_1,\ldots,x_N)\}$ is some open subset of $\mathbb{R}^{N+1}$.

$\mathcal{M}$ is the set of complex 2×2 matrices.

$\mathcal{A}(U\times\Omega)$ is the ring of $\mathcal{M}$-valued functions $\gamma=\gamma(k,x,t)$ of $(k,x,t)\in U\times\Omega$ which admit asymptotic expansions (AE) around $\{k_l\}_1^N$ and ∞:

$$\gamma \underset{k_l}{\sim} \sum_{n=-N'_l}^{\infty}\alpha_{ln}(k-k_l)^n \qquad \gamma \underset{\infty}{\sim} \sum_{n=-N'_\infty}^{\infty}\beta_n k^{-n} \tag{2}$$

where N'_l, $N'_\infty\in\mathbb{N}$ and the coefficients of the expansions are C^∞ functions of $(x,t)\in\Omega$.

$\mathcal{R}\subset\mathcal{A}(U\times\Omega)$ is the ring of $\mathcal{M}$-valued functions $\gamma=\gamma(k,x,t)$ which are C^∞ functions of $(x,t)\in\Omega$ and rational functions of k with possible poles at $\{k_l\}_1^N$ and ∞ only.

$\mathcal{A}(U\times\Omega)\rightarrow_P\mathcal{R}$ is the projection

$$P\gamma = \sum_{l=1}^{N}\sum_{n=-N'_l}^{-1}\alpha_{ln}(k-k_l)^n + \sum_{n=-N'_\infty}^{0}\beta_n k^{-n}. \tag{3}$$

$f_0(k,x,t)=\exp[-\mathrm{i}\sigma_3(\Sigma_{l=1}^N)x_l/(k-k_l)-\omega(k)t)]$ where $\omega(k)$ is a rational complex function of k with possible poles at $\{k_l\}_1^N$ only and such that $\omega(k)\rightarrow 0$ as $k\rightarrow\infty$. Let $\Sigma_{n=-\infty}^{N_l}\omega_{ln}/(k-k_l)^n$ be the Laurent expansion of $\omega(k)$ around k_l.

Let $\hat{\mathscr{F}}$ be a set of $\mathscr{M}$-valued functions $\hat{\varphi}(k, x, t)$ defined on $U\times\Omega$, and let $\mathscr{F}=\hat{\mathscr{F}}f_0$ be the corresponding set of functions $\varphi=\hat{\varphi}f_0$ with $\hat{\varphi}\in\hat{\mathscr{F}}$.

We will say that $\mathscr{F}$ is a MMAM if it satisfies the following properties.

Property 1. The elements of $\hat{\mathscr{F}}$ are C^∞ functions of (x, t).

Property 2. $\hat{\mathscr{F}}$ is a set of 'asymptotic' rational functions of k around $k_1, \ldots, k_N$ and ∞, with an 'asymptotic' unit element.

Property 3. $\mathscr{F}$ is a left $\mathscr{D}$ module, where $\mathscr{D}$ is the ring of linear differential operators generated by $\{\gamma\in\mathscr{R}, \partial_l=\partial/\partial x_l\ (1\leqslant l\leqslant N), \partial_t=\partial/\partial t\}$.

More precisely property 2 means that:

(i) $\hat{\mathscr{F}}\subset\mathscr{A}(U\times\Omega)$;

(ii) the restriction of the projection P to $\hat{\mathscr{F}}$ is one-to-one;

(iii) $\hat{\mathscr{F}}$ contains a privileged function $\hat{f}$ such that $\det\hat{f}\neq 0$ and $P\hat{f}=I$, i.e.

$$\begin{aligned} \hat{f}&=b_l+0(k-k_l) \qquad && k\to k_l \\ \hat{f}&=I+0(1/k) && k\to\infty. \end{aligned} \tag{4}$$

Each MMAM $\mathscr{F}$ is a left $\mathscr{R}$ module of dimension one generated by the basic element $f=\hat{f}f_0$. To prove this observe that property 3 implies that $\mathscr{F}$ is a left $\mathscr{R}$ module. Furthermore, given $\varphi\in\mathscr{F}$ it is easy to see that $P\hat{\varphi}=P(P\varphi f^{-1})\hat{f})$. Then from properties 2 and 3 we deduce $\varphi=P(\varphi f^{-1})f$, which completes the proof.

It is clear from property 3 that $\partial_l f$, $\partial_t f\in\mathscr{F}$. By taking into account the form of $f_0(k, t, x)$ and the fact that $P\hat{f}=I$ we find

$$\partial_l f=P\left(-\frac{\mathrm{i}}{k-k_l}f\sigma_3 f^{-1}\right)f \qquad l=1,\ldots,N \tag{5}$$

$$\partial_t f=P(\mathrm{i}\omega(k)f\sigma_3 f^{-1})f. \tag{6}$$

By replacing f with $\hat{f}f_0$ in (5) and (6) and using the AE of $\hat{f}$ at each $k_m\in\{k_l\}_1^N$ we get a system of $N+1$ non-linear compatible flows containing an infinite number of dependent variables consisting of the coefficients of the AE of $\hat{f}$ at the k_m. Our aim is to determine from these flows some systems of differential equations in the variables (x, t), involving only a finite number of dependent variables. To this end, we introduce

$$\hat{f}_l=b_l^{-1}\hat{f} \qquad r_l=\hat{f}_l\mathrm{i}\sigma_3\hat{f}_l^{-1} \qquad l=1,\ldots,N \tag{7}$$

which have AE of the form

$$\hat{f}_l=I+a_l(k-k_l)+\mathrm{O}((k-k_l)^2) \tag{8}$$

$$r_l=\sum_{n=0}^{\infty} r_{ln}(k-k_l)^n \qquad r_{l0}=\mathrm{i}\sigma_3 \qquad k\to k_l. \tag{9}$$

From (5) it follows at once that

$$\partial_l\hat{f}_l=v_l\hat{f}_l+\mathrm{i}\hat{f}_l\frac{\sigma_3}{k-k_l} \qquad v_l\equiv\mathrm{i}[\sigma_3, a_l]-\mathrm{i}\frac{\sigma_3}{k-k_l} \tag{10}$$

and as a consequence $\partial_l r_l=[v_l, r_l]$. At this point we notice that (7) implies $\mathrm{Tr}\, r_l=0$ and $\det r_l=1$, so that by following the procedure given in appendix A of [1] one concludes that the coefficients $r_{ln}(n\geqslant 0)$ in the AE (9) are polynomials in the matrix elements of

$$\mathrm{i}[\sigma_3, a_l]=\begin{pmatrix} 0 & -q_l \\ -s_l & 0 \end{pmatrix}$$

and their derivatives with respect to the coordinate x_l and can be obtained from recurrence relations.

Consider now equation (6). It can be written in the form

$$(\partial_t \hat{f})\hat{f}^{-1} = -b_l\,\omega(k) r_l b_l^{-1} + P(b_l \omega r_l b_l^{-1}). \tag{11}$$

By expanding both sides of (11) in powers of $(k-k_l)$ as $k \to k_l$ and by identifying the coefficients of the first two powers, one can prove that

$$\partial_t b_{lm} = \sum_{0\le n\le N_l} \omega_{ln} r_{ln} b_{lm} - \sum_{0\le n\le N_m} \omega_{mn} b_{lm} r_{mn} + \sum_{q=1}^{N} \sum_{p=1}^{N_q} \sum_{p\le n\le N_q} \omega_{qn}^{pm} b_{lq} r_{q,n-p} b_{qm} \tag{12}$$

$$\begin{aligned}\partial_t \mathrm{i}[\sigma_3, a_l] = {} & -\mathrm{i} \sum_{0\le n\le N_l} \omega_{ln}[\sigma_3, r_{l,n+1}] \\ & -\mathrm{i} \sum_{m\ne l} \sum_{p=1}^{N_m} \frac{p}{(k_l-k_m)^{p+1}} \left(\sum_{p\le n\le N_m} \omega_{mn}[\sigma_3, b_{lm} r_{m,n-p} b_{lm}^{-1}] \right)\end{aligned} \tag{13}$$

where we have set $b_{lm} = b_l^{-1} b_m$ and

$$\omega_{qn}^{pm} = \left(\frac{1-\delta_{qm}}{(k_m-k_q)^p} - \frac{1-\delta_{ql}}{(k_l-k_q)^p} \right) \omega_{qn}.$$

The same procedure applied to (10) instead of (11) yields for $l \ne m$

$$\mathrm{i}[\sigma_3, a_l] = (\partial_l b_{lm}) b_{lm}^{-1} + \frac{\mathrm{i}}{k_m - k_l} [\sigma_3, b_{lm}] b_{lm}^{-1} \tag{14}$$

$$\partial_l \mathrm{i}[\sigma_3, a_m] = \frac{1}{k_m - k_l} \left[\sigma_3, [\sigma_3, a_m] - \frac{1}{k_m - k_l} b_{lm}^{-1} \sigma_3 b_{lm} \right]. \tag{15}$$

The system (12)–(15) constitutes a system of differential relations involving the dependent variables $\{q_l, s_l, b_{lm}: l, m = 1, \ldots, N\}$ only, which arise as a consequence of (5) and (6). Observe that (12) and (13) are non-linear partial differential equations in $N+1$ dimensions, and that (14) and (15) are differential constraints which do not depend on the form of $\omega(k)$.

Equation (14) gives the variables q_l and s_l explicitly in terms of the matrix elements of b_{lm}; further it is not difficult to show that (13) derives from (12) and (14), so that (12) and (13) represent the same dynamical flow described in different variables. In fact, we may use (14) to express the coefficients r_{pq} in terms of b_{mn}, transforming (12) into a differential equation involving the matrices b_{mn} only. In addition; as a consequence of (14) and (15), these matrices are subject to the constraints

$$A_{lm} = A_{ln} \qquad \partial_l A_{mn} = \frac{\mathrm{i}}{k_l - k_m} [\sigma_3, (\partial_m b_{ml}) b_{ml}^{-1}] \tag{16}$$

where

$$A_{lm} \equiv (\partial_l b_{lm}) b_{lm}^{-1} + \frac{\mathrm{i}}{k_m - k_l} [\sigma_3, b_{lm}] b_{lm}^{-1}.$$

Thus we have eliminated the variables (q_l, s_l) from all the differential equations satisfied by the matrices b_{lm}. The final result is an evolution equation in $N+1$ dimensions together with the constraints (16).

Dealing with (13) is simpler than (12) in several respects. Firstly, one of the two types of terms which make up the right-hand side of (13) is given by the differential polynomials $[\sigma_3, r_{n+1}]$ in $\{q_l, s_l\}$ which occur in the AKNS hierarchy. Secondly, as we are going to see, the terms in (13) which depend on b_{lm} can be written as rational functions of (q_m, s_m) and their spatial derivatives, so that (13) becomes a differential equation involving the dependent variables (q_m, s_m) only. Let us denote

$$b_{lm} = \begin{pmatrix} b^1_{lm} & b^3_{lm} \\ b^2_{lm} & b^4_{lm} \end{pmatrix} \qquad r_{ln} = \begin{pmatrix} \xi_{ln} & \eta_{ln} \\ \gamma_{ln} & -\xi_{ln} \end{pmatrix}$$

then the b_{lm}-dependent terms in (13) can be written as

$$[\sigma_3, b_{lm} r_{mn} b^{-1}_{lm}] = \begin{pmatrix} 0 & u_{lmn} \\ v_{lmn} & 0 \end{pmatrix}$$
$$u_{lmn} = 2(d_{lm})^{-1}[(b^1_{lm})^2 \eta_{mn} - (b^3_{lm})^2 \gamma_{mn} - 2 b^1_{lm} b^3_{lm} \xi_{mn}] \tag{17}$$
$$v_{lmn} = 2(d_{lm})^{-1}[(b^2_{lm})^2 \eta_{mn} - (b^4_{lm})^2 \gamma_{mn} - 2 b^2_{lm} b^4_{lm} \xi_{mn}]$$

where $d_{lm} \doteqdot \det b_{lm} = \det f(k_m, x, t)/\det f(k_l, x, t)$ are complex constants since from (5) and (6) it follows that $\partial_l \det f = \partial_t \det f = 0$. In order to get rid of the quadratic products of the matrix elements of b_{lm} in (17) we use (15) which gives for $l \neq m$

$$4 b^1_{lm} b^2_{lm} = d_{lm}(k_l - k_m)^2 D_l(l, m) s_m \qquad 4 b^1_{lm} b^3_{lm} = -d_{lm}(k_l - k_m)^2 D_m(l, m) q_l$$
$$4 b^2_{lm} b^4_{lm} = -d_{lm}(k_l - k_m)^2 D_m(m, l) s_l \qquad 4 b^3_{lm} b^4_{lm} = d_{lm}(k_l - k_m)^2 D_l(m, l) q_m \tag{18}$$

where $D_l(m, n) \doteqdot \partial_l + 2\mathrm{i}/(k_m - k_n)$. Now (14) implies for $l \neq m$

$$D_l(l, m) b^2_{lm} = -s_l b^1_{lm} \qquad \partial_l b^4_{lm} = -s_l b^3_{lm}$$
$$D_l(m, l) b^3_{lm} = -q_l b^4_{lm} \qquad \partial_l b^1_{lm} = -q_l b^2_{lm} \tag{19}$$

which together with (18) leads to

$$4(b^1_{lm} b^4_{lm} + b^2_{lm} b^3_{lm}) = d_{lm}(k_l - k_m)^2 (s_l)^{-1} D_l(l, m) D_m(m, l) s_l \tag{20}$$

and to the constraint

$$(s_l)^{-1} D_l(l, m) D_m(m, l) s_l = (q_l)^{-1} D_l(m, l) D_m(l, m) q_l \qquad l \neq m. \tag{21}$$

We can then get the relations

$$b^1_{lm} b^4_{lm} = d_{lm}\left(\tfrac{1}{2} + \frac{(k_l - k_m)^2}{8 s_l} D_l(l, m) D_m(m, l) s_l\right)$$
$$(b^1_{lm})^2 = d_{lm}\left(-\tfrac{1}{2} - \frac{(k_l - k_m)^2}{8 s_l} D_l(l, m) D_m(m, l) s_l\right) \frac{D_m(l, m) q_l}{D_l(m, l) q_m} \tag{22}$$

and similar expressions for the other products $b^2_{lm} b^3_{lm}$, $(b^i_{lm})^2$ $(i = 2, 3, 4)$. It is now straightforward to rewrite the functions u_{lmn} and v_{lmn} as rational expressions in q_l, s_l and their spatial derivatives (note that these expressions are independent of the values of the d_{lm} so that (13) becomes an evolution equation for the scalar fields $\{(q_l, s_l), l = 1, \ldots, N\}$ in $(N+1)$ dimensions. In addition, the constraint (21) as well as the following ones are satisfied for $l \neq m$:

$$[D_l(m, l) q_m][D_l(l, m) s_m] = [D_m(l, m) q_l][D_m(m, l) s_l] \tag{23}$$
$$(q_l)^{-1} D_l(m, l) D_m(l, m) q_l = (q_m)^{-1} D_l(m, l) D_m(l, m) q_m \tag{24}$$
$$\frac{(k_l - k_m)^2}{16}\left(\frac{D_l(m, l) D_m(l, m) q_l}{q_l}\right)^2 - 1 = \frac{(k_l - k_m)^2}{4}[D_m(l, m) q_l][D_m(m, l) s_l]. \tag{25}$$

Equation (23) follows from the identity $(b^1_{lm}b^2_{lm})(b^3_{lm}b^4_{lm}) = (b^1_{lm}b^3_{lm})(b^2_{lm}b^4_{lm})$ and (25) from $(b^2_{lm}b^3_{lm})(b^1_{lm}b^4_{lm}) = (b^1_{lm}b^3_{lm})(b^2_{lm}b^4_{lm})$ while (24) is obtained by exchanging l and m in (20). Any previously defined MMAM $\mathcal{F}$ affords one solution to the system (S) of purely differential NE consisting of (13), (21) and (23)-(25). By varying $\omega(k)$ we generate a hierarchy of solvable systems (S).

The system (S) admits interesting reductions which can be characterised through the analysis of MMAM. To illustrate this point we will outline the following reduction. Provided $\{k_l\}_1^N$ are real numbers and $\bar{\omega}(\bar{k}) = \omega(k)$, each MMAM $\mathcal{F}$ satisfying $\sigma_1\bar{\varphi}(\bar{k})\sigma_1 \in \mathcal{F}$ for all $\varphi \in \mathcal{F}$, provided a basic element f satisfying $\sigma_1\bar{f}(\bar{k})\sigma_1 = f(k)$. In this way, we also obtain the reductions $s_l = \bar{q}_l$, $b^3_{lm} = \overline{b^2_{lm}}$, $b^4_{lm} = \overline{b^1_{lm}}$. Thus by taking $\omega(k) = \sum_{l=1}^N A_l/(k-k_l)^2$, where $\{k_l\}_1^N$ and $\{A_l\}_1^N$ are real numbers, (S) yields the system $(1a)$-$(1d)$.

We can construct MMAM by considering $\bar{\partial}$ equations

$$\frac{\partial\varphi}{\partial\bar{k}}(k) = \varphi(k)r(k) \qquad k \in \mathbb{C} - \{k_l\}_1^N \tag{26}$$

with 2×2 matrix-valued distributions $r(k)$ going to zero fast enough as $k \to k_l$ $(l = 1, \ldots, N)$ and $k \to \infty$ [1]. The set $\mathcal{F}$ of solutions $\varphi(k, x, t)$ of (26) such that φf_0^{-1} admit AE of the form (2) is a MMAM provided some reasonable assumptions on $r(k)$ are satisfied. There are other ways of characterising MMAM. An important one is the following. Let q_0 and q_0' be two different complex numbers in $\mathbb{C} - \{k_l\}_1^N$ and $\mathbb{C}^2 = M \oplus N$ be a non-trivial decomposition of $\mathbb{C}^2$ into two subspaces M, N. We define $\mathcal{F}$ as the set of functions $\varphi(k, x, t)$ analytic in $\mathbb{C} - (\{k_l\}_1^N \cup \{q_0\})$, with φf_0^{-1} having at most poles at the k_l and ∞, and such that:

(i) φ has at most a simple pole at q_0 and the corresponding residue satisfies $R(M) = \{0\}$;

(ii) the value S of φ at q_0' satisfies $S(N) = \{0\}$.

It is easy to prove that $\mathcal{F}$ is a MMAM. Moreover by reproducing arguments given in [1] one finds that the corresponding basic solution f represents the pure one-soliton solution of the system (S) consisting of the $(N+1)$-dimensional NE (13) in general (or $(1a)$ if $\omega(k) = \sum_{l=1}^N [A_l/(k-k_l)^2]$) and of the non-linear constraints (21)-(25) (or $(1b)$-$(1d)$). The generalisation of this procedure for characterising multisoliton solutions is straightforward. It should be observed that these soliton solutions are not truly $(N+1)$ dimensional but rather are wave solitons.

References

[1] Jaulent M, Manna M and Martinez Alonso L 1988 Multi-series Lie groups and asymptotic modules for characterising and solving integrable models *Preprint* Montpellier P M 88/23

[2] Jaulent M, Manna M and Martinez Alonso L 1988 Scalar-bipolar asymptotic modules for solving (2+1)-dimensional non-linear evolution equations with constraints *Phys. Lett.* A in press

[3] Jaulent M, Manna M and Martinez Alonso L 1988 Matrix-bipolar asymptotic modules for solving (2+1)-dimensional non-linear evolution equations with constraints *J. Phys. A: Math. Gen.* **21** L1019

[4] Jaulent M, Manna M and Martinez Alonso L 1988 An integrable (2+1)-dimensional generalisation of the Volterra model *J. Phys. A: Math. Gen.* **21** L719

[5] Ablowitz M J, Kaup D J, Newell A C and Segur H 1974 *Stud. Appl. Math.* **53** 294